

Solving the N -Queens Problem with Local Search

Enrico Schumann
es@enricoschumann.net

This vignette provides example code for a combinatorial problem: the *N-Queens Problem*.

1 The problem

The goal is to place N queens on a chess-board of size $N \times N$ in such a way that no queen is attacked. A queen may move vertically, horizontally and on a diagonal. So whenever there is more than one queen on any row, column or diagonal, the position is invalid. To solve the problem with a Local Search (LS), we need three components:

1. a way to represent a solution (i.e. a position on the chessboard);
2. a way to evaluate such a solution;
3. and, since we use a LS, a method to modify a solution.

We start by attaching the package and fixing a seed.

```
> library("NMOF")  
> set.seed(134577)
```

2 Representing a solution

Since on any row there cannot be more than one queen, we may store a position as a vector of columns on which the queens are placed. (In chess, rows would be called ranks and columns would be files, but we prefer matrix terminology.) Thus, a candidate solution p (p for position) could look as follows:

```
> N <- 8          ## board size  
> p <- sample.int(N) ## a random solution  
> data.frame(row = 1:N, column = p)
```

row	column
1	8
2	7
3	3
4	1
5	6
6	2
7	5
8	4

Or (a very bad solution):

```
> p <- rep(1, N)  
> data.frame(row = 1:N, column = p)
```

row	column
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1

We will also want to visualise a position, for which we write the function `print_board`.

```
> print_board <- function(p, q.char = "Q", sep = " ") {
  n <- length(p)
  row <- rep("-", n)
  for (i in seq_len(n)) {
    row_i <- row
    row_i[p[i]] <- q.char

    cat(paste(row_i, collapse = sep))
    cat("\n")
  }
}
```

```
> print_board(p)
```

```
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
```

3 Evaluating a solution

We need to compute on what row, column, diagonal (top left to bottom right) or reverse diagonal (top right to bottom left) a queen stands. Rows and columns are simple; we label the diagonals as follows.

```
> mat <- array(NA, dim = c(N,N)) ## diagonals
> for (r in 1:N)
  for (c in 1:N)
    mat[r,c] <- c - r
```

```
> mat
```

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]
[1,]	0	1	2	3	4	5	6	7
[2,]	-1	0	1	2	3	4	5	6
[3,]	-2	-1	0	1	2	3	4	5
[4,]	-3	-2	-1	0	1	2	3	4
[5,]	-4	-3	-2	-1	0	1	2	3
[6,]	-5	-4	-3	-2	-1	0	1	2
[7,]	-6	-5	-4	-3	-2	-1	0	1
[8,]	-7	-6	-5	-4	-3	-2	-1	0

```
> mat <- array(NA, dim = c(N,N)) ## reverse diagonals
> for (r in 1:N)
  for (c in 1:N)
    mat[r,c] <- c + r - (N + 1)
```

```
> mat
```

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]
[1,]	-7	-6	-5	-4	-3	-2	-1	0
[2,]	-6	-5	-4	-3	-2	-1	0	1
[3,]	-5	-4	-3	-2	-1	0	1	2
[4,]	-4	-3	-2	-1	0	1	2	3
[5,]	-3	-2	-1	0	1	2	3	4
[6,]	-2	-1	0	1	2	3	4	5
[7,]	-1	0	1	2	3	4	5	6
[8,]	0	1	2	3	4	5	6	7

Note that for reverse diagonals, the $N + 1$ would not be necessary; it serves only to shift the diagonal labels so that the main diagonal is zero.

Thus for a given solution p , we know the row, column, diagonal and reverse diagonal for each queen. We define the quality of a solution by the number of attacks that happen: for a valid solution, that number should be zero.

```
> n_attacks <- function(p) {
  ## more than one Q on a column?
  sum(duplicated(p)) +

  ## more than one Q on a diagonal?
  sum(duplicated(p - seq_along(p))) +

  ## more than one Q on a reverse diagonal?
  sum(duplicated(p + seq_along(p)))
}
> n_attacks(p)
```

```
[1] 7
```

4 Changing a solution

A given position may be modified by picking one row randomly and then moving the queen there to the left or right. We allow for moves up to $step$ squares, which we set to 3 in the example.

```
> neighbour <- function(p) {
  step <- 3
  i <- sample.int(N, 1)
  p[i] <- p[i] + sample(c(1:step, -(1:step)), 1)

  if (p[i] > N)
    p[i] <- 1
  else if (p[i] < 1)
    p[i] <- N
  p
}
```

```
> print_board(p)
```

```
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
```

```
> print_board(p <- neighbour(p))
```

```
Q - - - - -
Q - - - - -
- - - Q - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
```

```
> print_board(p <- neighbour(p))
```

```
Q - - - - -
Q - - - - -
- - - Q - - -
Q - - - - -
Q - - - - -
- - - - - Q
Q - - - - -
Q - - - - -
```

5 Solving the model

We use three different LS methods: a ‘classical’ Stochastic Local Search (LSopt), Threshold Accepting (TAopt) and Simulated Annealing (SAopt).

```
> p0 <- rep(1, N) ## or a random initial solution: p0 <- sample.int(N)
> print_board(p0)
```

```
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
Q - - - - -
```

```
> sol <- LSopt(n_attacks, list(x0 = p0,
                              neighbour = neighbour,
                              printBar = FALSE,
                              nS = 10000))
```

```
Local Search.
Initial solution: 7
Finished.
Best solution overall: 0
```

```
> print_board(sol$xbest)
```

```
- Q - - - - -
- - - - - Q -
- - Q - - - -
- - - - - Q -
- - - - - Q
- - - - - Q
- - - - - Q
Q - - - - -
- - - Q - - -
```

```
> sol <- TAopt(n_attacks, list(x0 = p0,
                              neighbour = neighbour,
                              printBar = FALSE,
                              nS = 1000))
```

```
Threshold Accepting

Computing thresholds ... OK
Estimated remaining running time: 0.185 secs

Running Threshold Accepting ...
Initial solution: 7
```

```
Finished.  
Best solution overall: 0
```

```
> print_board(sol$xbest)
```

```
- - - - Q - - -  
- - - - - Q -  
- Q - - - - -  
- - - Q - - - -  
- - - - - Q  
Q - - - - -  
- - Q - - - -  
- - - - Q - -
```

```
> sol <- SAopt(n_attacks, list(x0 = p0,  
                               neighbour = neighbour,  
                               printBar = FALSE,  
                               nS = 1000))
```

```
Simulated Annealing.  
  
Calibrating acceptance criterion ... OK  
Estimated remaining running time: 0.19 secs.  
  
Running Simulated Annealing ...  
Initial solution: 7  
Finished.  
Best solution overall: 0
```

```
> print_board(sol$xbest)
```

```
- - - - Q - - -  
- - - - - Q -  
- Q - - - - -  
- - - Q - - - -  
- - - - - Q  
Q - - - - -  
- - Q - - - -  
- - - - Q - -
```

References

Manfred Gilli, Dietmar Maringer, and Enrico Schumann. *Numerical Methods and Optimization in Finance*. Elsevier, 2011.